Subspace Methods For Harmonic Models

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Task 1 Estimate the PSD of a real signal by various methods

\[ x(n) = \sin(\omega_1 n + \phi_1) + 0.5 \sin(\omega_2 n + \phi_2) + v(n) \]

with \( \omega_1 = 0.2\pi, \omega_2 = 0.3\pi, \phi_i \sim U(0, 2\pi), i = 1,2 \). And \( v(n) \sim N(0,1) \)

a) Periodogram
As in project 1, here set the data length \( N = 100 \) and the FFT size \( L = 1024 \).

b) Pisarenko Harmonic Decomposition
Each ‘sin’ consists of two analytical frequencies: positive and negative. So the signal space dimension is \( P = 4 \). To get the eigenvector of the correlation matrix can be done directly by the singular value decomposition (SVD) of the data matrix: In ESPRIT, we use the eigenvectors corresponding to the signal space, while here in PHD and MUSIC, we use the eigenvectors corresponding to the noise space. We should pay attention that in different methods the data matrices are constructed with different \( M \).

In PHD, noise space is one-dimensional, so I select the eigenvector corresponding to the smallest eigenvalue of the data matrix with \( M = P + 1 \) as the noise subspace basis.

c) MUSIC
The noise space in MUSIC is not necessarily one-dimensional as in PHD. It can average the multi-dimensional noise to lessen the effect of spurious peak.

There’re three combinations of parameters for MUSIC shown in the right column of the figure below. When \( M = 5 = P + 1 \), it’s essentially the same as PHD.

We can see that MUSIC can estimate the frequency better, especially with a bigger window \( M \). By PHD, once the estimation error in correlation matrix occurs, or once the noise space eigenvector of the data matrix is inaccurate, there’s the spurious peak, as is clearly seen from the PHD or MUSIC (\( M = 5 \)) above. But MUSIC with \( M > 5 \) can average the noise out to some extent. And a longer sample length will be helpful, too.
Task 2 Estimate the PSD of analytical signal by various methods

\[ x(n) = \sum_{p=1}^{P} \alpha_p e^{j\pi p} a_p + w(n) \quad n = 1, \ldots, N \]

with \( \alpha_p = a_p e^{i\phi_p} \quad a_1 = 1, \quad a_2 = 0.5, \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.25 \). And \( w(n) \sim N(0, \sigma^2) \)

a) Periodogram

Now that there’re two asymmetric analytical frequencies, the normalized frequency is shown totally from \(-0.5\) to \(0.5\).

b) MUSIC

The signal space dimension is \( P = 2 \).

c) ESPRIT

ESPRIT can only tell us the exact location of frequencies, or it’s the line spectrum. In order to compare it, I plot it as vertical lines in the same plot with the MUSIC.

Parameters combinations:

1. \( M = 10, \quad N = 50, \quad \sigma_2 = 1 \)

![Figure 2](image)

Here the frequency estimation by ESPRIT are 0.0983 and 0.1246 Hz, which are quite close to the theoretical values 1.00 and 1.25 Hz. The Periodogram estimates are close to the real values, too. But the MUSIC peaks shift away a little bit.
Here the frequency estimation by ESPRIT are 0.0985 and 0.1253 Hz. The three methods are all estimating well with sufficiently long samples.
3. $M = 10, N = 50, \sigma_2 = 5$

When the noise is too big to be overwhelming, the Periodogram method cannot give the estimation on signal frequencies any more. The MUSIC and ESPRIT can still give a rough estimation of the frequency on the bigger signal, but their performance is not perfect to show the small sinusoidal component, either.
4. \( M = 10, \ N = 50, \ \sigma_2 = 1 \) but set \( P = 3 \)

![Figure 5](image)

When there's over-modeling, the frequency estimations by ESPRIT are 0.1046, 0.1576 and 0.2977 Hz: the model mismatch seriously degrades the performance of ESPRIT. But for MUSIC, the effect of over-modeling is weaker in the sense that the two largest peaks are still roughly corresponding the real signal frequencies, but the third peak is still a big problem. As to the Periodogram method, since it is a non-parametric method, over-modeling won't affect it at all.
5. \( M = 10, \ N = 50, \ \sigma_2 = 1 \) but set \( P = 1 \)

![Plot for Task2_5](image)

When there’s under-modeling, the frequency estimations by ESPRIT is 0.1014 Hz: the model mismatch seriously degrades the performance of both MUSIC and ESPRIT. There’s only one peak present in their estimations, even though this peak is corresponding to the largest signal frequency. As to Periodogram method, since it is a non-parametric method, under-modeling won’t affect it.
Conclusions

Nonparametric Periodogram method is robust since it doesn’t need any model information, but the concentration is not good enough. Parametric methods such as PHD, MUSIC and ESPRIT can utilize the system model information, thus when there’s no model-mismatching, they can provide better performance than nonparametric method. Among the parametric methods, I think MUSIC with big snapshot size $M$ is the best method. And if practical condition permits, a longer sample will be more helpful.

As to programming, even though I have constructed two subroutines for PHD and MUSIC, in practice we only need to set $M=P+1$ in MUSIC to realize PHD.

Further, in getting the noise subspace basis, there’s indeed a difference between the result from SVD of data matrix and eigen-decomposition of correlation matrix $R_x$, and even in getting $R_x$, there’re several different ways, but I have tried these different ways in MUSIC and they are quite similar as following for task1:

![Figure 7: Appendix: Comparison of different ways in getting noise space basis](image)

This simulation can ease us somehow to choose whichever way to get the noise space basis. Even in getting the signal space basis in ESPRIT, we can use the eigen-decomposition of $R_x$ and choose those eigenvectors corresponding to the signals: I have tried this way, and the simulation shows that the corrmtx(***, ‘modified’) is a preferred way because this mode can give us an unbiased estimation on $R_x$. 