Time-Frequency Signal Representations

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Before the discussion of time-frequency representation, I’d like to recall the long journey for me to get to appreciate Dr. Fourier’s brilliant idea on frequency representation of signals.

I first met the Fourier series for periodic continuous signals when I was in college, and later I learned the Fourier Transform for aperiodic continuous signals. At that time, I felt that it’s so incredible for lots of frequency components to accurately strengthen or cancel each other at certain time point to compose whatever signals along time. If there were a world existing in frequency domain, the people there might feel in the same way to admire the fabulous time components that could strengthen and cancel each other at certain frequency point to compose whatever signals along frequency. Time Domain and Frequency Domain are such an amazing dual.

Then here in the classes of DSP I and DSP II, we’ve talked a lot on Discrete-time Fourier transform (DTFT), and Discrete Fourier Transform (DFT).

DTFT is the Fourier transform of a discrete time sequence. Here we can regard the time sequence as a dual of Fourier series. Now that Fourier series is corresponding to periodic continuous time representation, time sequence is naturally corresponding to a periodic continuous frequency representation: as we’ve discussed in DSP I, the DTFT is indeed periodic but with a fixed period $\pi$.

DFT is the sampled version of DTFT. In time domain, we can sample a continuous signal into a time sequence, and it’s the sampling theorem that guides the analysis of the time sequence. Then in the frequency domain, we can use a similar sampling theorem to analyze the frequency sequence. Frequency aliasing and time aliasing are the dual concepts in this analysis.

By DTFT and DFT, all continuous representations can be sampled and then processed by their corresponding discrete sequences, so we can deal with the Fourier transforms by the computers easily and efficiently, and the above comparison of time domain and frequency domain can help us interpret the discrete data which either input to or output from the computers.

Till now we’ve limited our representation of signals in either time domain or in frequency domain, can we cross the two domains’ boundary, and let the two representations coexist?

1. **What is time-frequency representation? Why is it needed?**

   In some particular situations, we can extract certain information about one domain by the representation in the other domain.

   For example, in Frequency Modulation, there’s only one single frequency at each time point and we can differentiate the signal’s instantaneous phase to get the instantaneous frequency $f_i$.

   $$f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \arg x_i$$

   In dual, in our FIR filter design, we purposely delay some desirable impulse response
to make it causal, thus the resultant Linear Time Invariant system have a group delay and we can differentiate the phase of its frequency response to get the group delay \([1]\).

\[
t_x(f) = \frac{1}{2\pi} \frac{d}{df} \arg X(f)
\]

But what if we have a signal with two or more frequency components at a single time point or if there’re two or more time delay contributions at one frequency point? Then above methods are no longer able to extract the exact information. In another word, above methods may give us an average idea, but these methods cannot show us how many components exist and what they are.

In order to understand what frequency components exist at certain time point, and during what time certain frequency component exists, we need to consider a surface over the time-frequency plane. We denote such time-frequency representation (TFR) of a signal \(x(t)\) as \(T_x(t, f)\). In practice, the TFR can be linear, quadratic or even nonlinear. Now you may wonder why \(T_x(t, f)\) isn’t unique, let’s first consider two extreme cases in Fourier transform:

\[
F \left( e^{j\omega t} \right) = 2\pi \delta(w - w_0)
\]

\[
F \left( \delta(t - t_d) \right) = e^{-j\omega t_d}
\]

The Fourier pair of a frequency impulse is an infinite-duration sinusoidal wave; and the Fourier pair of a time impulse has infinite bandwidth. i.e. we cannot localize a signal at certain time-frequency point at all.

This is a vivid illustration on radar uncertainty principle \([4]\). Assume that the effective time duration of a signal is \(\alpha\), and the effective bandwidth is \(\beta\), then \(\alpha\beta \geq \pi\). So we cannot interpret \(T_x(t, f)\) as an exact measurement of the components at each time-frequency point, so the time-frequency representation cannot be uniquely defined. Therefore in defining different \(T_x(t, f)\), what we can do is only to make them possess certain desirable mathematical properties for the convenience of analysis in different applications.

In short, we import \(T_x(t, f)\) because we want to represent a signal in more detail, which is especially useful in dealing with wide-band signals \([3]\) or non-stationary signals \([4]\) where the conventional Fourier transform is not sufficient.

In practice, linear and quadratic TFRs are most widely used, so we shall concentrate on them as following.

2. Linear time-frequency representations

"Linear transform" means that it complies with the superposition law.

\[
x(t) = c_1x_1(t) + c_2x_2(t)
\]

\[
\Rightarrow T_x(t, f) = c_1T_{x_1}(t, f) + c_2T_{x_2}(t, f)
\]

There’re two important linear time-frequency representations: Short-time Fourier Transform and the Wavelet Transform.

2.1 Short-time Fourier Transform (STFT)

The STFT is also called Time-Dependent Fourier Transform \([2]\), as defined by \([1]\):
in which the “analysis window” \( \gamma^* (t - \tau) \) is centered at \( t \). From the definition of STFT, it’s easy to verify that STFT is linear, and it preserves frequency shift and time shift up to a modulation:

\[
\tilde{x}(t) = x(t)e^{j2\pi t_0} \Rightarrow STFT^\gamma_x(t, f) = STFT^\gamma_x(t, f - f_0)
\]

and

\[
\tilde{x}(t) = x(t-t_d) \Rightarrow STFT^\gamma_x(t, f) = STFT^\gamma_x(t-t_d, f)e^{-j2\pi tuf}
\]

The STFT is just a local spectrum of the signal around the analysis time. The analysis window \( \gamma^* (t - \tau) \) will affect the STFT in the same way as the windowing effect in DFT.

We can also express STFT by the signal and window spectra \[^1\] : 

\[
STFT^\gamma_x(t, f) = e^{-j2\pi ft} \int \left[ X(v)\Gamma^*(v - f) \right] e^{j2\pi tv} dv \tag{2.1.2}
\]

Thus the STFT can be looked upon as the inverse Fourier transform of the windowed spectrum \( X(v)\Gamma^* (v - f) \). In natural, \( \Gamma^* (f) \) is a low pass filter.

Further, given the STFT, we can use following synthesis formula to get back \( x(t) \)^[1]:

\[
x(t) = \int \int STFT^\gamma_x(\tau, v)g(t-\tau)e^{j2\pi vt} d\tau dv \tag{2.2}
\]

in which, \( g(t-\tau) \) is the synthesis window satisfying

\[
\int g(t)\gamma^* (t)dt = 1 \tag{2.3}
\]

We may have different analysis window \( \gamma^* (t - \tau) \), thus the synthesis window \( g(t-\tau) \) is not unique. In fact even if we use the same analysis window, there’re multiple choices for the synthesis window as long as they satisfy formula (2.3).

In discrete form,

\[
STFT^\gamma_x(nT, kF) = \int \left[ x(\tau)\gamma^* (\tau - nT) \right] e^{-j2\pi kF \tau} d\tau \tag{2.4}
\]

\[
x(t) = \sum_{n} \sum_{k} STFT^\gamma_x(nT, kF)g(t-nT)e^{j2\pi kFt} \tag{2.5}
\]

s.t. \( \frac{1}{F} \sum_{n} g(t+k \frac{1}{F} - nT)\gamma^* (t-nT) = \delta(k) \quad \forall t \) \tag{2.6}

where \( T > 0 \) and \( F > 0 \) are the sampling periods for the time and frequency variables, and \( n \) and \( k \) are integers. In simulations, we need to use the discrete form of the transforms, but I won’t list the discrete forms one by one any more.

2.2 The Wavelet Transform (WT)

The Wavelet Transform (WT) is defined by \[^1\] : 

\[
WT^\gamma_x(t, f) = \int x(\tau)\sqrt{f/f_0} \gamma^* \left( f/f_0 (\tau - t) \right) d\tau \tag{2.7}
\]
in which the analyzing wavelet $\gamma(t)$ is a band-pass filter centered around $t = 0$ in time domain and around $f_0$ in frequency domain.

The WT preserves time shift and time scaling, but it doesn’t preserve frequency shift:

$$\tilde{x}(t) = x(t-t_a) \Rightarrow WT_x^{(\gamma)}(t,f) = WT_x^{(\gamma)}(t-t_a,f)$$

Set analysis scale $a = \frac{f_0}{f}$, then

$$\tilde{x}(t) = \sqrt{a} |x(at)| \Rightarrow WT_x^{(\gamma)}(t,f) = WT_x^{(\gamma)}(at, \frac{f}{a})$$

The quality factor of the band-pass filter $Q$ is the ratio of the analysis frequency and the bandwidth, which is independent of the analysis frequency, so the WT is also called constant-$Q$ analysis \[^1\]. In another word, the bandwidth of the band-pass filter is proportional to the analysis frequency, and the duration of the band-pass filter is inversely proportional to the analysis frequency. As regard to the resolution, the higher the analysis frequency, the better the time resolution, but the poorer the frequency resolution.

Further, by the similar synthesis formula as in (2.2) & (2.3) or in discrete form as in (2.5) & (2.6), we can get back $x(t)$ from its WT.

### 2.3 The comparison of STFT and WT

After the definition and property statement of STFT and WT in 2.1, 2.2, we can get a rough idea on how STFT and WT act on signal $x(t)$. In common, they both use analysis windows to balance the time resolution and the frequency resolution in order to represent the signal in the time-frequency plane, and just because of the windowing, the spectrum are both broadened even for the impulses.

The same as the windowing effect in DSP, here the window selection will also affect the shape of STFT and WT.

In comparison, their analysis windows are quite different thus they have different mathematical properties. In STFT, the window itself is a low-pass filter, and the window won’t change its shape or bandwidth after time shift or frequency shift. But in WT, the window is a band-pass filter, and its bandwidth is inversely proportional to the analysis central frequency, so WT does not preserve frequency shift.

### 3. Quadratic time-frequency representations

If we square the signal or its Fourier transform, we can get its energy/power, this information is quite useful in many applications, thus we also want to define some quadratic form of the time-frequency representations and try to make the marginal integration of the quadratic time-frequency representation equal the energy/power \[^1\]:

$$\int_{f} T_x(t,f) df = P_x(t) = |x(t)|^2$$

$$\int_{t} T_x(t,f) dt = P_x(f) = |X(f)|^2$$

(3.1)

Meantime, the power can be obtained from the correlation functions, so we want to preserve similar relation in quadratic time-frequency representation, too \[^1\].
We expect the quadratic TFRs to satisfy above marginal properties, but we should be aware that not all definitions could fulfill this requirement. We need to trade off between these desirable marginal properties and other mathematical requirements.

Further note that the quadratic transforms are no longer linear, so they won’t comply with the superposition law any more. In general

\[
T_x(t, f) = c_1 x_1(t) + c_2 x_2(t)
\]

\[
\Rightarrow T_x(t, f) = |c_1|^2 T_{x_1}(t, f) + |c_2|^2 T_{x_2}(t, f) + c_1 c_2^* T_{x_1 x_2}(t, f) + c_2 c_1^* T_{x_2 x_1}(t, f)
\]

i.e. the cross-term interference will naturally occur, which is characteristic for quadratic time-frequency representations.

3.1 Spectrogram and Scalogram

Intuitively, if we square the linear time-frequency representations such as STFT or WT, we can interpret the quadratic form roughly as energy.

Somebody has already named the square of STFT and WT as spectrogram and scalogram respectively [1].

\[
\text{SPEC}_{x}(t, f) = |STFT_x^{(y)}(t, f)|^2
\]

\[
\text{SCAL}_x^{(y)}(t, f) = |WT_x^{(y)}(t, f)|^2
\]

Recall that the discrete version of the analysis windows of STFT and WT are practically band-limited (for it has significant amplitude only during certain bandwidth periodically) and duration-finite, so if the components are far apart in time-frequency space, i.e. there’s no overlap on the auto terms, then the cross term interference is essentially null.

3.2 Wigner Distribution (WD)

Aside from the direct square of the linear time-frequency representation, there’re many other quadratic forms of the time-frequency representation. Wigner Distribution (WD) is such a definition with many desirable mathematical properties [1].

\[
W_{x,y}^{(x,y)}(t, f) = \int_{\tau} x(t + \tau)x^*(t - \tau)e^{-j2\pi f \tau}d\tau
\]

\[
= \int_{\nu} X(f + \nu)X^*(f - \nu)e^{j2\pi \nu \tau}d\nu
\]

It’s easy to verify that the auto term is always real-valued. The WD preserves time shift and frequency shift, and it satisfies the marginal properties in (3.1). Loosely the WD can be interpreted as the distribution of signal power over the time-frequency plane, but due to the radar uncertainty principle mentioned earlier, we shouldn’t interpret WD as the power spectral density distribution point-wisely. And here’s a practical illustration on this point: we all know that power can never be negative, but it’s shown that WD can locally assume negative values.

When defining the linear time-frequency representation STFT and WT, you may have
noticed that because of the windowing, the time and frequency resolutions are broadened, so are their squares Spectrogram and Scalogram. In contrast, WD doesn’t use any window, so WD preserves the time and frequency concentration.

Despite the good concentration, WD has a drawback in practical applications, that is, the interference terms in WD are always present, no matter how far away the two signals are apart, thus the interference terms may overlap with other signal components. From the definition, we can see that the Interference terms are characteristic in WD: Even for one consistent signal, there’s interference term between its sub-parts.

The interference terms in WD are oscillatory pretty quick. The direction of oscillation is perpendicular to the line connecting the two signals. And the farther the two signals are apart, the quicker the oscillation.

3.3 Ambiguity Function (AF)

We know that the autocorrelation function and the power spectral density (PSD) function are a Fourier pair, then what will the Fourier pair of WD look like? WD’s Fourier pair is Ambiguity-Function (AF): 

$$A_{x,y}(\tau, v) = \int \int W_{x,y}(t, f) e^{j2\pi(\tau f - \tau v)} \, dt \, df$$ (3.4)

AF satisfies the marginal property (3.2), so AF is indeed like a correlation function. Here’s the direct definition on AF:

$$A_{x,y}(\tau, v) = \int x(t + \frac{\tau}{2}) y^* (t - \frac{\tau}{2}) e^{-j2\pi \nu v} \, dt$$

$$= \int X(f + \frac{\nu}{2}) Y^* (f - \frac{\nu}{2}) e^{j2\pi \tau f} \, df$$ (3.5)

AF can be loosely interpreted as a joint time-frequency correlation function. The signal’s AF is naturally centered at the origin. And the maximum value of an auto AF occurs at the origin and equals the signal’s energy.

Then concerning the interference terms in AF, they are located symmetrically around the lag points in the \((\tau, \nu)\) plane. ‘Lag’ here means the difference between two signal components.

AF has been used a lot in estimating the distance and velocity of a moving target in radar systems. Physically, the radar AF represents the output of a matched filter, which is matched to a reference target of equal Radar Cross Section (RCS). The AF evaluated at the origin is matched perfectly to the signal reflected from the target of interest, or the nominal target. The AF evaluated at other time-frequency points represents the returns from other range and Doppler shifts which are different from those for the nominal target.

Radar designers also use the radar AF as a means of studying different waveforms. The AF can provide insight about how the different radar waveforms may be suitable for the various radar applications.

3.4 Smoothed Pseudo-WD (SPWD)

As we discussed in 3.2, the interference terms in WD are sometimes troublesome, but they are oscillatory, so we can utilize certain low-pass filter to smooth the WD in order to attenuate the interference.
The smoothed pseudo-WD (SPWD) is defined as [1]:

$$SPWD_x^{(g,H)}(t,f) = \int_{\tau} \int_{\nu} g(t-\tau)H(f-\nu)W_x(\tau,\nu) d\tau \, dv$$

(3.6)

In which the window $g(t)$ and $H(f)$ independently determine the smoothing in time and in frequency respectively.

Windowing will broaden the resolution, so we need to balance the interference attenuation and the time-frequency concentration by appropriately selecting the two windows $g(t)$ and $H(f)$.

4. Applications and Simulations

After discussing the definitions and properties for different TFRs, here’s a simple list on their applications.

<table>
<thead>
<tr>
<th>TFR</th>
<th>Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>STFT/ Spectrogram</td>
<td>Spectral Estimation, Speech pitch and formant analysis, Complex modulation, Dynamic compression</td>
</tr>
<tr>
<td>WT/ Scalogram</td>
<td>WT differs to STFT in analyzing higher frequencies with better time resolution but poorer frequency resolution</td>
</tr>
<tr>
<td>WD</td>
<td>Useful analysis tool in quantum mechanics, acoustics, seismic and mechanical vibrations</td>
</tr>
<tr>
<td>AF</td>
<td>Estimate the distance &amp; velocity of a moving target in radar system</td>
</tr>
</tbody>
</table>

In discussing the definition of the TFRs, basically we only consider their continuous forms. Then in the simulation, we can use our DFT knowledge to do the transforms by discrete processing.

All the plots and illustrations are shown in the Appendix.

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References:


Appendix: Simulation Results

Let's first look at the STFT and WT of a simple signal with two time impulses and two frequency impulses:

Signal 1 in simulation

\[ x(t) = \delta(t-t_1) + \delta(t-t_2) + e^{i2\pi f_1} + e^{i2\pi f_2} \]

with \( t_1 = 20s \quad t_2 = 80s \quad f_1 = 0.2f_s \quad f_2 = 0.4f_s \)

Its Fourier spectrum is

\[ X(f) = e^{-i2\pi f_1} + e^{-i2\pi f_2} + \delta(f-f_1) + \delta(f-f_2) \]

![Fig. 1](image1.png)

STFT of signal 1 in surface format

The four peaks are right on the cross points of the four impulses

![Fig. 2](image2.png)

STFT of signal 1 in contour format

In Journals, the contour format is used a lot, so I show the same STFT as in Fig. 1 in contour format here. It's clearly seen that the spectrum of even the impulses are broadened and the degree of broadening is all the same even for different analysis time and different analysis frequency.
Comparing it with the STFT of signal 1, they are common in that the spectrum are both broadened, but WT differs from STFT by that the degree of broadening is different for different analysis time and different analysis frequency. The higher the analysis frequency, the better the time resolution, but the poorer the frequency resolution.

Now in order to study the effect of cross term interference for quadratic TFRs, let’s consider another signal composed of two components, which are far away apart in time-frequency plane.

\[
x(t) = \begin{cases} 
  e^{j2\pi f_1 t} & 0 < t < t_1 \\
  e^{j2\pi f_2 t} & t_2 < t < t_3 \\
  0 & \text{otherwise}
\end{cases}
\]

with \( t_1 = 32s, t_2 = 96s, t_3 = 128s \)
\( f_1 = 0.2f_s, f_2 = 0.45f_s \)

It’s clearly seen from the plots that the cross term interference is essentially null for both the Spectrogram and the Scalogram, if the two components are far away apart.
Fig. 6  WD of signal 1 in contour format

This plot shows that the WD spectrum of the four impulses in signal 1 is very thin, or WD has a much better concentration than Spectrogram or Scalogram not only in time but also in frequency. But between the four ‘lines’ representing the four pulses, the scattered ‘dots’, representing the cross term interference, are omnipresent and not weak.

Fig. 7  WD of signal 2 in contour format  Fig. 8  WD of signal 2 in surface format

Fig. 7 shows that even for signal 2, the WD still has cross term interference in the midway. The same plot in surface format is shown in Fig. 8. We can see that the cross term interference is oscillating very fast and the WD can assume negative values locally, so we cannot interpret WD as PSD point-wisely due to the radar uncertainty principle.

Fig. 9  AF of signal 2

AF is like the correlation functions: it concentrates at the origin, and there’re always interference terms at the lag points. Here the “lag” means the time & frequency difference between two components.