Angle-only Filtering in 3D using Modified Spherical and Log Spherical Coordinates

Mahendra Mallick  
Propagation Research Associates, Inc.  
Marietta, GA 30066, USA  
mahendra.mallick@gmail.com

Sanjeev Arulampalam  
Maritime Operations Division, DSTO  
Edinburgh SA 5111, Australia  
Sanjeev.Arulampalam@dsto.defence.gov.au

Lyudmila Mihaylova  
School of Computing and Communications  
Lancaster University, LA1 4WA, UK  
mila.mihaylova@lancaster.ac.uk

Yanjun Yan  
ARCON Corporation  
Waltham, MA, 02451, USA  
yanjun@arcon.com

Abstract—This paper considers the angle-only filtering problem in 3D using bearing and elevation angle measurements from a single maneuvering sensor. We develop continuous-discrete extended Kalman filter (EKF) based algorithms using modified spherical coordinates (MSC) and log spherical coordinates (LSC), where the dynamic and measurement models are described in continuous and discrete times, respectively. The predicted state estimate and covariance are calculated numerically by integrating a joint vector differential equation. Numerical results show that, while the discrete-time Cartesian EKF outperforms the proposed algorithms for highly accurate measurements, the new algorithms show superior performance as the measurement accuracy decreases. EKF-MSC and EKF-LSC have comparable performance for the examples examined.

Keywords: Angle-only filtering in 3D, Nonlinear stochastic differential equations, Modified spherical coordinates (MSC), Log spherical coordinates (LSC), Continuous-discrete filtering.

I. INTRODUCTION

The angle-only filtering problem in 3D using bearing and elevation angles from a single maneuvering sensor is the counterpart of the bearing-only filtering problem in 2D [1], [5], [17]. This problem arises in passive ranging using an infrared search and track (IRST) sensor, passive sonar, and passive radar in the presence of jamming [6]. A great deal of research has been conducted for the bearing-only filtering problem in 2D - see for e.g. [1], [5], [17], and the references therein. However, the number of publications for the angle-only filtering problem in 3D is relatively small [19], [18], [15], [10], [16].

The angle-only filtering problem in 3D using modified spherical coordinates (MSC) [19] or log spherical coordinates (LSC) is much harder than the bearing-only filtering problem in 2D using modified polar coordinates (MPC) [1], [17] or log polar coordinates (LPC) [8]. This is due to the fact that discretization of the dynamic model for the 3D case using MSC or LSC is not as straightforward as for the 2D case using MPC [1] or LPC [8]. Therefore, we use the continuous-discrete extended Kalman filter (EKF) using MSC and LSC.

Stallard [19] first extended the method from [12] and [14] to three dimensions and proposed the MSC. There are three important differences between our approach and Stallard’s approach, as follows:

Firstly, we use standard conventions [17] in defining the coordinate frames for the tracker and sensor, and coordinates for MSC and LSC. In [19], the coordinate frames, bearing and elevation angle are defined in a non-standard way.

Secondly, we clearly show the equivalence of the nearly constant velocity model (NCVM) using Cartesian state vector and stochastic differential equations using the MSC or LSC. This is missing in [19]. We illustrate this by first presenting the NCVM in Cartesian coordinates with a continuous white noise acceleration process [3], together with its power spectral density matrix, expressed in the tracker coordinate frame (T frame). Then we derive the first order nonlinear stochastic differential equations for MSC and LSC corresponding to the NCVM in Cartesian coordinates.

Thirdly, and most importantly, we present a new derivation of the differential equation for the predicted covariance following the Brownian motion process [13], when the nonlinear time evolution function depends on the target state and continuous-time process noise. In most common nonlinear stochastic differential equations, the continuous-time process noise appears as an additive term as in chapter 6 of [9]. The nonlinear time evolution function for MSC or LSC depends on the target state, continuous-time process noise and ownship acceleration in the sensor frame (S frame). The predicted state estimate and covariance are integrated numerically and jointly to provide better numerical accuracy. This is a key contribution of our paper. In [19], the state transition matrix, discrete-time process noise covariance matrix, and discrete time predicted covariance matrix are derived using the approximation that the relative geometry between the target and ownship changes relatively slowly.

The paper is organized as follows. Section II introduces notations and the coordinate frames with respect to the target and ownship. Section III presents the dynamic models of the target and ownship, and measurement model with respect to Cartesian relative state. Sections IV and V present the MSC
and LSC respectively, whereas the designed filter is described in Section VI. Results are shown in Section VII and finally the conclusions are summarized in Section VIII.

II. TRACKER COORDINATE FRAME (T FRAME)

We define the tracker coordinate frame for which the X, Y, and Z axes are along the local east, north, and upward directions, respectively. The origin of the T frame has geodetic longitude \( \lambda_0 \), geodetic latitude \( \phi_0 \) and geodetic height \( h_0 \).

The 3D angle-only tracking problem is considered under the following assumptions:

1) We estimate the state of a non-maneuvering target using bearing (\( \beta \)) and elevation (\( \epsilon \)) angle measurements.
2) The target motion is described by a NCVM in Cartesian coordinates [3] in 3D. The state of the target is defined in the T frame.
3) The motion of the ownship or sensor is deterministic, i.e., non-random. We know the state of the ownship precisely.

The ownship performs maneuvers so that the target state becomes observable. Since we use standard conventions for coordinate frames and angle variables, the rotational transformation \( T^S_T \) from the T frame to the S frame is defined differently from that in [19].

III. SYSTEM DYNAMICS AND MEASUREMENT MODELS

A. Dynamic Model for State Vector and Relative State Vector in Cartesian Coordinates

The Cartesian states of the target and ownship are defined, respectively, by \( x^t := \begin{bmatrix} x^t, y^t, z^t \end{bmatrix}^T \) and \( x^o := \begin{bmatrix} x^o, y^o, z^o \end{bmatrix}^T \). The relative state vector of the target is defined by

\[
x := x^t - x^o.
\]

Let \( [x, y, z, \hat{x}, \hat{y}, \hat{z}] \) denote the components of \( x \) in the T frame. Then the state vector can be written in the form:

\[
x := \begin{bmatrix} x & y & z & \hat{x} & \hat{y} & \hat{z} \end{bmatrix} = \begin{bmatrix} x^t - x^o & y^t - y^o & z^t - z^o & \hat{x}^t & \hat{y}^t & \hat{z}^t \end{bmatrix}.
\]

Let \( r^T \) denote the range vector of the the target from the sensor in the T frame. Then \( r^T \) is defined by

\[
r^T := \begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} x^t - x^o & y^t - y^o & z^t - z^o \end{bmatrix}.
\]

The bearing and elevation angles are defined in Figure 1. The slant range (or simply range) is defined as

\[
r := \|r^T\| = \sqrt{x^2 + y^2 + z^2}.
\]

The range vector can be expressed in terms of range (\( r \)), bearing (\( \beta \)) and elevation (\( \epsilon \)) by

\[
r^T = r \begin{bmatrix} \cos \epsilon \sin \beta \\ \cos \epsilon \cos \beta \\ \sin \epsilon \end{bmatrix}, \quad \beta \in [0, 2\pi], \quad \epsilon \in [-\pi/2, \pi/2].
\]

The ground range is defined by

\[
\rho := \sqrt{x^2 + y^2} = r \cos \epsilon, \quad \rho > 0.
\]

![Fig. 1. Definition of tracker coordinate frame (T frame), bearing \( \beta \in [0, 2\pi] \) and elevation angle \( \epsilon \in [-\pi/2, \pi/2] \).](image)

B. Measurement Model

The measurement model for the bearing and elevation angles using the relative Cartesian state vector \( x_k \) is

\[
z_k = h(x_k) + n_k,
\]

where

\[
h(x_k) = \begin{bmatrix} \beta_k \\ \epsilon_k \end{bmatrix} = \begin{bmatrix} \tan^{-1}(x_k, y_k) \\ \tan^{-1}(z_k, \rho_k) \end{bmatrix},
\]

\[
n_k \sim N(n_k; 0, R_k), R_k := \text{diag}([\sigma_\beta^2, \sigma_\epsilon^2]).
\]

C. Continuous-time Dynamic Model of the Target

The continuous-time linear dynamic model for the target state is described by [3], [9]

\[
\frac{dx^t(t)}{dt} = Ax^t(t) + w^t(t),
\]

where \( w^t(t) \) is a zero-mean continuous-time Gaussian white noise with power spectral density matrix \( Q^t(t) \)

\[
E\{w^t(t)\} = 0,
\]

\[
E\{w^t(t)(w^t(\tau))'\} = \delta(t-\tau)Q^t(t),
\]

where \( \delta \) is the Dirac delta function.

Let \( \{a_x, a_y, a_z\} \) denote the continuous-time zero-mean white noise acceleration processes with power spectral densities \( \{q_x, q_y, q_z\} \) respectively.

For the nearly constant velocity motion (NCVM) in 3D [3]

\[
A = \begin{bmatrix} 0_3 & I_3 \\ 0_3 & 0_3 \end{bmatrix},
\]

where \( 0_3 \) is the zero matrix of dimension \( 3 \times 3 \) and \( I_3 \) is the \( 3 \times 3 \) identity matrix.

Define the target and ownship accelerations in the T frame, respectively, by

\[
a^T(t) := \begin{bmatrix} a_x^T(t), a_y^T(t), a_z^T(t) \end{bmatrix}' = \begin{bmatrix} \dot{x}^t(t), \dot{y}^t(t), \dot{z}^t(t) \end{bmatrix}',
\]

\[
a^T(t) := \begin{bmatrix} \ddot{x}^t(t), \ddot{y}^t(t), \ddot{z}^t(t) \end{bmatrix}'.
\]
\[
\mathbf{a}_o^\mathbf{T}(t) := \left[ \dot{x}^o(t), \dot{y}^o(t), \dot{z}^o(t) \right]', \quad (14)
\]
Then the acceleration of the target relative to the ownship is defined by
\[
\mathbf{r}^\mathbf{T}(t) := \left[ \dot{x}(t), \dot{y}(t), \dot{z}(t) \right]' = \mathbf{a}^\mathbf{T}(t) - \mathbf{a}_o^\mathbf{T}(t). \quad (15)
\]
The process noise in the T frame is described by
\[
\mathbf{w}^\mathbf{T}(t) = \begin{bmatrix} 0 \\ \mathbf{a}^\mathbf{T}(t) \end{bmatrix},
\]
where \( E\{ \mathbf{a}^\mathbf{T}(t) \} = 0, E\{ \mathbf{a}^\mathbf{T}(t) \mathbf{a}^\mathbf{T}(\tau) \} = \delta(t - \tau)\mathbf{Q}^\mathbf{T} \), and \( \mathbf{Q}^\mathbf{T} = \text{diag}(q_x, q_y, q_z) \). Then the process noise power spectral density matrix is
\[
\mathbf{Q}^\mathbf{T}(t) = \begin{bmatrix} 0_3 & 0_3 \\ 0_3 & \mathbf{Q}^\mathbf{T} \end{bmatrix}. \quad (17)
\]

IV. STOCHASTIC DIFFERENTIAL EQUATIONS FOR MODIFIED SPHERICAL COORDINATES

Instead of using \( \beta \) as a component of the MSC, following Stallard’s convention \([2], [19]\), we use \( \omega \) where
\[
\omega(t) := \beta(t) \cos \epsilon(t). \quad (18)
\]
Let \( \zeta(t) \) denote the logarithm of range \( r(t) \).
\[
\zeta(t) := \ln r(t). \quad (19)
\]
Then
\[
r(t) = \exp \zeta(t). \quad (20)
\]
Differentiating (19) with respect to time, we get
\[
\dot{\zeta}(t) = \frac{\dot{r}(t)}{r(t)}. \quad (21)
\]
The relative state vector of the target in MSC is defined by \([2], [19]\)
\[
\mathbf{\xi}(t) := \left[ \xi_1(t) \quad \xi_2(t) \quad \xi_3(t) \quad \xi_4(t) \quad \xi_5(t) \quad \xi_6(t) \right]' = \left[ \omega(t) \quad \dot{\zeta}(t) \quad \beta(t) \quad \epsilon(t) \quad \frac{1}{r(t)} \right]' \quad (22)
\]
We follow the approach of Stallard \([19]\) to derive the stochastic differential equations for MSC. We summarize the key steps of the derivation here. Detailed steps of the derivation are presented in Appendix A.

1) Derive expressions for the relative velocity \((\dot{x}, \dot{y}, \dot{z})\) and relative acceleration \((\ddot{x}, \ddot{y}, \ddot{z})\) as functions of range, bearing, and elevation and their derivatives in the T frame.
2) Define a S frame such that the Z axis of the S frame is along the range vector.
3) Using the bearing and elevation angles, calculate the rotational transformation matrix \(\mathbf{T}_S^T\) from the T frame to the S frame.
4) Transform the relative acceleration \(\mathbf{\ddot{r}}^\mathbf{T}\) in the T frame to the S frame to yield \(\mathbf{\ddot{r}}^\mathbf{S}\), which are functions of MSC.
5) Equate \(\mathbf{\ddot{r}}^\mathbf{S}\) to the difference of the target acceleration (white noise acceleration) and ownship acceleration, expressed in the S frame. This produces the desired stochastic differential equations for the MSC.

The rotational transformation matrix \(\mathbf{T}_S^T\) is obtained by a (3,2) Euler sequence with Euler angles \((\phi = \pi/2 - \beta, \theta = \pi/2 - \epsilon)\). The Z axis of the S frame is along the range vector. We emphasize that the measurements \((\beta, \epsilon)\) are not defined in the S frame, but in the T frame. The S frame is defined for obtaining the decoupled stochastic differential equations for the MSC or LSC. Thus, the transformation matrix \(\mathbf{T}_S^T\) is
\[
\mathbf{T}_S^T = \begin{bmatrix} \sin \epsilon \sin \beta & \sin \epsilon \cos \beta & -\cos \epsilon \\
\cos \epsilon \sin \beta & \cos \epsilon \cos \beta & 0 \\
\cos \epsilon \sin \beta & \sin \epsilon \cos \beta & \sin \epsilon \sin \beta \end{bmatrix}. \quad (23)
\]
Detailed derivations for the stochastic differential equations in MSC are presented in Appendix A and the desired equations are
\[
\frac{d\mathbf{\xi}}{dt} = \mathbf{f}^{\mathbf{MSC}}(\mathbf{\xi}, \mathbf{a}^S, \mathbf{a}_o^S), \quad (24)
\]
where
\[
\mathbf{f}^{\mathbf{MSC}}(\mathbf{\xi}, \mathbf{a}^S, \mathbf{a}_o^S) := \begin{bmatrix} \xi_1(\xi_2 \tan \xi_5 - 2\xi_6) - \xi_6(\mathbf{a}_o^S - \mathbf{\dot{a}}_o^S) \\
-2\xi_2 \xi_3 \xi_5 - \xi_5^2 \xi_5 - \xi_6(\mathbf{a}_o^S - \mathbf{\dot{a}}_o^S) \\
\xi_1^2 + \xi_2^2 - \xi_3^2 + \xi_6(\mathbf{a}_o^S - \mathbf{\dot{a}}_o^S) \\
\xi_1 / \cos \xi_5 \\
\xi_2 \\
-\xi_3 \xi_6 \end{bmatrix}, \quad (25)
\]
\[
\mathbf{a}^S(t) = \mathbf{T}_S^T \mathbf{a}^\mathbf{T}(t), \quad (26)
\]
\[
\mathbf{a}_o^S(t) = \mathbf{T}_S^T \mathbf{a}_o^\mathbf{T}(t). \quad (27)
\]
We note that \(\mathbf{a}^S(t)\) is the continuous-time zero-mean white noise acceleration process defined in the S frame with the power spectral density matrix
\[
\mathbf{Q}^S = \mathbf{T}_S^T \mathbf{Q}^\mathbf{T}(\mathbf{T}_S^T)' \quad (28)
\]
According to our assumptions, \(\mathbf{a}_o^S(t)\) is the deterministic acceleration of the ownship defined in the S frame.

V. STOCHASTIC DIFFERENTIAL EQUATIONS FOR LOG SPHERICAL COORDINATES

The first five components of the LSC are the same as those of MSC in (22). Following [8], we use the sixth component of the LSC as the logarithm of the range as defined in (19). Thus, the relative state vector of the target in LSC is defined by
\[
\mathbf{\eta}(t) := \left[ \eta_1(t) \quad \eta_2(t) \quad \eta_3(t) \quad \eta_4(t) \quad \eta_5(t) \quad \eta_6(t) \right]' = \left[ \omega(t) \quad \dot{\zeta}(t) \quad \beta(t) \quad \epsilon(t) \quad \frac{1}{r(t)} \right]' \quad (29)
\]
Since the first five components of the MSC and LSC are the same, differential equations for the first five components of LSC can be easily obtained from those of MSC by noting from (20) that
\[
\xi_6 = 1/r = \exp(-\eta_6) \quad (30)
\]
We also have
\[
\eta_6 = \dot{\zeta} = \eta_3. \quad (31)
\]
Thus the stochastic differential equations in LSC are given by
\[
\frac{d\mathbf{\eta}}{dt} = \mathbf{f}^{\mathbf{LSC}}(\mathbf{\eta}, \mathbf{a}^S, \mathbf{a}_o^S), \quad (32)
\]
where
\[ f^{LSC}(\eta, a^S, a^S_0) := \begin{bmatrix} \eta_1 \tan \eta_3 - 2\eta_3 - \exp(-\eta_6)(a^S_y - y^S_0) \\ -2\eta_2 \eta_3 - \eta_1^2 \tan \eta_3 - \exp(-\eta_6)(a^S_x - x^S_0) \\ \eta_1^2 + \eta_2^2 + \exp(-\eta_6)(a^S_z - z^S_0) \\ \eta_1 / \cos \eta_2 \\ \eta_2 \\ \eta_3 \end{bmatrix} . \tag{33} \]

VI. EXTENDED Kalman FILTER USING MSC AND LSC

First, consider nonlinear state estimation using MSC. State estimation using LSC will follow a similar approach. The cumulative measurement vector at time \( t_k \) is defined by
\[ Z_k := \{ z_1, z_2, ..., z_k \} . \tag{35} \]

A. Filter Initialization

Suppose the initial range prior is \( r \sim N(\bar{r}, \sigma_r^2) \) where \( \bar{r} \) and \( \sigma_r^2 \) are the mean and variance, respectively, of the initial range. Likewise, assume that the prior for the speed is \( s \sim N(\bar{s}, \sigma_s^2) \). Furthermore, suppose we have some prior knowledge of target heading (\( \alpha, \gamma \)), where \( \alpha \) and \( \gamma \) are the bearing and elevation components, respectively, of the heading vector. Let this prior be Gaussian given by \( \alpha \sim N(\bar{\alpha}, \sigma_\alpha^2) \) and \( \gamma \sim N(\bar{\gamma}, \sigma_\gamma^2) \).

Given the above prior statistics and the initial bearing and elevation measurements \( (\beta_1, \epsilon_1) \), the Cartesian state vector and its covariance can be initialized similar to [17]
\[ \dot{x}_1 = \begin{bmatrix} \bar{r} \cos \epsilon_1 \sin \beta_1 \\ \bar{r} \cos \epsilon_1 \cos \beta_1 \\ \bar{s} \cos \gamma \sin \bar{\alpha} - \bar{z}^1_1 \\ \bar{s} \cos \gamma \cos \bar{\alpha} - \bar{y}^1_1 \\ \bar{s} \sin \gamma - \bar{z}^1_1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} P_{\text{pos}} & 0 & 0_{3 \times 3} \\ 0_{3 \times 3} & P_{\text{vel}} \end{bmatrix}, \tag{36} \]

where \((\bar{x}^1_0, \bar{y}^1_0, \bar{z}^1_0)\) is the ownship velocity vector at time index 1. The elements of the matrices \( P_{\text{pos}} \) and \( P_{\text{vel}} \) can be obtained using a Taylor series approximation as in [13].

Suppose the transformation from Cartesian to MSC is
\[ \xi = \mathcal{Y}(x) . \tag{37} \]

Then, given \( \dot{x}_1 \) and \( P_1 \) in Cartesian coordinates, and the transformation to MSC in (37), the state and covariance in MSC can be initialized using an unscented transform [17]. The LSC initialization is carried out in a similar manner.

B. Prediction of State Estimate and Covariance

This section presents the algorithm for the predicted state estimate and covariance equations.
\[ \hat{\xi}(t) := E\{\xi(t)|Z_{k-1}\}, \quad t_{k-1} \leq t < t_k, \tag{38} \]
\[ \hat{\xi}(t) := \xi(t) - \hat{\xi}(t), \tag{39} \]
\[ P(t) := E\{(\xi(t) - \hat{\xi}(t))(\xi(t) - \hat{\xi}(t))^T|Z_{k-1}\}, \quad t_{k-1} \leq t < t_k. \tag{40} \]

Taking the conditional expectation of (24) and interchanging the order of expectation and differentiation on the left hand side of the equation, we get
\[ \dot{\xi}(t) = \hat{f}^{MSC}(\xi, a^S, a^S_0), \quad t_{k-1} \leq t < t_k, \tag{41} \]
where
\[ \hat{f}^{MSC}(\xi, a^S, a^S_0) := E\{f^{MSC}(\xi, a^S, a^S_0)|Z_{k-1}\}, \quad t_{k-1} \leq t < t_k. \tag{42} \]

For simplicity in notation, we use \( w = a^S \). In Appendix B, we show that the differential equations for the predicted state and predicted covariance are
\[ \dot{\xi}(t) \approx \hat{f}^{MSC}(\xi(t), 0, a^S_0), \quad t_{k-1} \leq t < t_k, \tag{43} \]
\[ \dot{P}(t) \approx F(\xi(t))P(t) + P(t)F'(\xi(t)) + G(\xi(t))Q^S G'(\xi(t)), \tag{44} \]
where \( t_{k-1} \leq t < t_k \) and
\[ F(\xi(t)) := \frac{\partial \hat{f}^{MSC}(\xi(t), w, a^S_0)}{\partial \xi(t)}|_{\xi(t)=\hat{\xi}(t), w=0} , \tag{45} \]
\[ G(\xi(t)) := \frac{\partial \hat{f}^{MSC}(\xi(t), w, a^S_0)}{\partial w(t)}|_{\xi(t)=\hat{\xi}(t), w=0} . \tag{46} \]

The predicted state estimate and covariance equations for LSC are similarly derived. Expressions for the derivative matrices \( F \) and \( G \) for MSC and LSC are presented in Appendix B.

The first order nonlinear differential equation (43) can be integrated using the Matlab function ‘ode45.’ Examining (44), we observe that the derivative matrices \( F \) and \( G \) depend on the state estimate \( \xi(t) \). Therefore, it is necessary to integrate (43) and (44) together by defining an augmented vector which includes \( \xi(t) \) and the distinct components of \( P(t) \). Since \( \xi(t) \) has \( n \) elements and \( P(t) \) is a symmetric matrix with \( n(n+1)/2 \) distinct components, the augmented vector has \( n(n + 3)/2 \) elements.

C. Measurement Update Step

The measurement model for the bearing and elevation angles \( \beta_k, \epsilon_k \) using MSC is
\[ z_k = H \xi_k + n_k, \tag{47} \]
\[ H := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} . \tag{48} \]

The measurement model for LSC is essentially the same as for MSC, except that \( \xi_k \) in (47) is replaced by \( \eta_k \). Measurement updated state estimate and covariance are given by the standard KF update algorithm [3].
VII. NUMERICAL SIMULATIONS AND RESULTS

Our testing scenario is extended from [7] to 3D. The application is for an airborne aircraft intercept mission in which the goal is to detect and establish an accurate track on the target at long range using the angle-only measurements from an IRST sensor. The target moves primarily in a plane parallel to the XY-plane at a nearly constant height of 9 km with a NCVM in 3D. The motion of the ownship is deterministic. The ownship moves in a plane parallel to the XY-plane at a fixed height of 10 km.

Initial ground range \( \rho_1 \), bearing \( \beta_1 \), target height \( z^t_1 \), and sensor height \( z^s_1 \) are shown in Table I. Then the initial elevation angle \( \epsilon_1 \) can be calculated. Table I also presents initial speed \( s_1 \), course \( c_1 \), and \( Z \) component of target velocity \( \dot{z}^t_1 \).

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho_1 ) (km)</td>
<td>138.0</td>
</tr>
<tr>
<td>( \beta_1 ) (deg)</td>
<td>45.0</td>
</tr>
<tr>
<td>( \epsilon_1 ) (deg)</td>
<td>-0.415</td>
</tr>
<tr>
<td>( z^t_1 ) (km)</td>
<td>9.0</td>
</tr>
<tr>
<td>( z^s_1 ) (km)</td>
<td>10.0</td>
</tr>
<tr>
<td>( s_1 ) (m/s)</td>
<td>297.0</td>
</tr>
<tr>
<td>( c_1 ) (deg)</td>
<td>-135.0</td>
</tr>
<tr>
<td>( \dot{z}^t_1 ) (m/s)</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Cartesian components of the target initial position and velocity calculated from parameters in Table I are given in Table II.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^t_1 ) (m)</td>
<td>97,580.736</td>
</tr>
<tr>
<td>( y^t_1 ) (m)</td>
<td>97,580.736</td>
</tr>
<tr>
<td>( z^t_1 ) (m)</td>
<td>9,000,000</td>
</tr>
<tr>
<td>( \dot{x}^t_1 ) (m/s)</td>
<td>-210.011</td>
</tr>
<tr>
<td>( \dot{y}^t_1 ) (m/s)</td>
<td>-210.011</td>
</tr>
<tr>
<td>( \dot{z}^t_1 ) (m/s)</td>
<td>0.000</td>
</tr>
</tbody>
</table>

The ownship motion consists of constant velocity (CV) motion and coordinated turn (CT). The profile of the ownship motion is presented in Table III and has seven segments with the motion sequence CV, CT, CV, CT, CT, CV, CT, CV, as illustrated in Figure 2. The target and ownship trajectories are shown in Figure 3.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>( \Delta t ) (s)</th>
<th>( \Delta \phi ) (rad)</th>
<th>Motion Type</th>
<th>( \omega^{\phi} ) (rad/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>([\tau_0, \tau_1] = [0, 15])</td>
<td>15</td>
<td>0</td>
<td>CV</td>
<td>0</td>
</tr>
<tr>
<td>([\tau_1, \tau_2] = [15, 31])</td>
<td>16</td>
<td>-( \pi /4 )</td>
<td>CT</td>
<td>-( \pi /64 )</td>
</tr>
<tr>
<td>([\tau_2, \tau_3] = [31, 45])</td>
<td>12</td>
<td>0</td>
<td>CT</td>
<td>0</td>
</tr>
<tr>
<td>([\tau_3, \tau_4] = [45, 55])</td>
<td>10</td>
<td>( \pi /2 )</td>
<td>CT</td>
<td>( \pi /64 )</td>
</tr>
<tr>
<td>([\tau_4, \tau_5] = [55, 85])</td>
<td>10</td>
<td>0</td>
<td>CV</td>
<td>0</td>
</tr>
<tr>
<td>([\tau_5, \tau_6] = [85, 102])</td>
<td>16</td>
<td>-( \pi /4 )</td>
<td>CT</td>
<td>-( \pi /64 )</td>
</tr>
<tr>
<td>([\tau_6, \tau_7] = [102, 210])</td>
<td>108</td>
<td>0</td>
<td>CV</td>
<td>0</td>
</tr>
</tbody>
</table>

The power spectral densities \( (q_x, q_y, q_z) \) of the zero-mean white acceleration process noise along the X, Y, and Z axes of the T frame are \((0.01, 0.01, 0.0001)\) \( \text{m}^2 \text{s}^{-3} \), respectively. The measurement sampling time interval is 1.0 s.

We used \((0.001, 0.005, 0.015)\) radian for the bearing and elevation measurement error standard deviations, representing high, medium, and low measurement accuracies, respectively. RMSEs for Cartesian position and velocity calculated from 500 Monte Carlo simulations are presented in Figures 4-9. Figures 4-5 show that, when the measurement accuracy is high (0.001 radian), the Cartesian EKF performs the best after 50 seconds, when the transient effect had decayed. When the measurement accuracy is medium (0.005 radian) or low (0.015 radian), then Figures 6-9 show that EKF-MSC and EKF-LSC perform better than the CEKF. The accuracies of the EKF-MSC and EKF-LSC are nearly the same for all measurement accuracies.
Fig. 4. Root mean square position error with 0.001 radian standard deviation for bearing and elevation.

Fig. 5. Root mean square velocity error with 0.001 radian standard deviation for bearing and elevation.

Fig. 6. Root mean square position error with 0.005 radian standard deviation for bearing and elevation.

Fig. 7. Root mean square velocity error with 0.005 radian standard deviation for bearing and elevation.

VIII. Conclusions

We have developed new algorithms for the angle-only filtering problem in 3D using EKF-MSC and EKF-LSC. The dynamic model for the target is represented by a continuous-time nonlinear stochastic differential equation and the measurement model is a discrete-time linear measurement model. For this continuous-discrete filtering problem, we integrate the equation of the predicted state estimate and covariance numerically and simultaneously without making a first order approximation for the state transition matrix as is done in [19].

The performance of the EKF-MSC and EKF-LSC are compared with that of the baseline algorithm, Cartesian EKF. Results from Monte Carlo simulations show that when the measurement accuracy for the bearing and elevation measurements is high (0.001 radian), the CEKF performs better than the EKF-MSC and EKF-LSC. However, when the measurement accuracy for the bearing and elevation measurements is medium (0.005 radian) or low (0.015 radian), EKF-MSC and EKF-LSC perform better than the CEKF. Our results show that EKF-MPC and EKF-LPC have nearly the same accuracy for the examples examined.

EKF-MSC and EKF-LSC are computationally more intensive due to integration of the joint nonlinear differential equations for the predicted state estimate and covariance. The CPU times of the EKF-MSC and EKF-LSC are about 90 times higher than that of the CEKF. Our future work will consider more detailed evaluation of these three filtering algorithms under various operating conditions.

Appendix A. Derivations for Stochastic Differential Equations in MSC and LSC

Differentiating (4) with respect to the time, we get

\[ \dot{\mathbf{r}}_x = \dot{r} \cos \epsilon \sin \beta - r \dot{\epsilon} \sin \epsilon \sin \beta + r \dot{\beta} \cos \epsilon \cos \beta, \]  
(49)

\[ \dot{\mathbf{r}}_y = \dot{r} \cos \epsilon \cos \beta - r \dot{\epsilon} \sin \epsilon \cos \beta - r \dot{\beta} \cos \epsilon \sin \beta, \]  
(50)

\[ \dot{\mathbf{r}}_z = \dot{r} \sin \epsilon + r \dot{\epsilon} \cos \epsilon. \]  
(51)

Differentiating (49), (50), and (51) with respect to the time, we get

\[ \ddot{\mathbf{r}}_x = (\ddot{r} - r \dot{\epsilon}^2 - r \dot{\beta}^2) \cos \epsilon \sin \beta - (2 \dot{r} \dot{\epsilon} + r \ddot{\epsilon}) \sin \epsilon \sin \beta \]  
\[ + (2 \dot{r} \dot{\beta} + r \ddot{\beta}) \cos \epsilon \cos \beta - 2r \dot{\epsilon} \dot{\beta} \sin \epsilon \cos \beta, \]  
(52)
The equation of motion (15) for the relative acceleration in bearing and elevation.

\[
\dddot{\mathbf{r}} - \dddot{\boldsymbol{\beta}} \cos \epsilon \sin \epsilon - r \dddot{\epsilon}.
\]

Simplification of (63) and substitution in the X component of (55) gives

\[
\dot{\epsilon} = -2 \zeta \dot{\epsilon} - \omega^2 \tan \epsilon - (a_y^S - \dot{r}_y^S)/r.
\]

Simplification of (58) gives

\[
\dot{r}_z^S = \ddot{r} - r \dot{\varphi}^2 - r \ddot{\beta}^2 \cos^2 \epsilon.
\]

Writing (61) using MSC, we get

\[
\dot{\xi}_1 = \xi_1 (\xi_2 \tan \xi_3 - 2 \xi_4) - \xi_6 (a_y^S - \dot{y}_y^S).
\]

Writing (64) using MSC, we get

\[
\dot{\xi}_2 = -2 \xi_2 \xi_3 - \xi_2^2 \tan \xi_5 - \xi_6 (a_z^S - \dot{z}_z^S).
\]

Simplification of (66) and substitution in the Z component of (55) gives

\[
\ddot{\xi}_3 = \xi_3^2 + \xi_2^2 - \xi_3^2 + \xi_6 (a_z^S - \dot{z}_z^S).
\]

Using (18), we get

\[
\dot{\xi}_4 = \dot{\beta} = \omega / \cos \epsilon = \xi_1 / \cos \xi_5.
\]

From definitions

\[
\xi_5 = \dot{\epsilon} = \xi_2,
\]

\[
\xi_6 = (1/\varrho) = -(1/\varrho) (\dot{r}/r) = -\xi_3 \xi_6.
\]

The stochastic differential equations derived in this appendix are represented by the nonlinear vector differential equation (24). Stochastic differential equations for LSC can be derived using similar approach.

### Appendix B. Expressions for Derivative Matrices F and G in MSC and LSC

\[
\mathbf{F}^{\text{MSC}}(\mathbf{e}, \mathbf{a}^S, \mathbf{a}^S_0) =
\begin{bmatrix}
\xi_2 \tan \xi_3 - 2 \xi_3 \tan \xi_5 - 2 \xi_4 \tan \xi_5 - 2 \xi_1 \xi_2 \sec^2 \xi_5 - (a_y^S - \dot{y}_y^S) \\
-2 \xi_2 \tan \xi_5 - 2 \xi_3 - 2 \xi_2 - \xi_4 \xi_3 \sec^2 \xi_5 - (a_y^S - \dot{y}_y^S) \\
2 \xi_1 \sec \xi_5 \xi_3 \sec \xi_5 \\
0 \xi_1 \xi_2 \sec \xi_5 \\
0 \xi_1 \xi_3 \sec \xi_5 \\
0 0 0 -\xi_6 \\
0 0 0 -\xi_6 \\
0 0 0 -\xi_6 \\
0 0 0 -\xi_6
\end{bmatrix}
\]

\[
\mathbf{G}^{\text{MSC}}(\mathbf{e}, \mathbf{a}^S, \mathbf{a}^S_0) =
\begin{bmatrix}
0 -\xi_6 \\
-\xi_6 0 0 \\
0 0 \xi_6 \\
0 0 0 \xi_6 \\
0 0 0 0
\end{bmatrix}
\]
The nonlinear system dynamics model in continuous-time is

\[ \dot{x}(t) = f(x(t), b(t)), \] (77)

where \( \dot{b}(t) = w(t) \) and \( b(t) \) is a Brownian process [13]. For \( \epsilon \rightarrow 0 \)

\[ (x(t + \epsilon) - x(t))/\epsilon = f(x(t), (b(t + \epsilon) - b(t))/\epsilon), \] (78)

\[ x(t + \epsilon) = x(t) + \epsilon f(x(t), (b(t + \epsilon) - b(t))/\epsilon). \] (79)

Let \( w(t) := (b(t + \epsilon) - b(t))/\epsilon \). The variance of the increment \( b(t + \epsilon) - b(t) \) is proportional to \( \epsilon \) and \( w(t) \sim \mathcal{N}(0, Q(t)/\epsilon) \). The first-order Taylor series expansion of (79) about the point \( (\bar{x}, 0) \) where \( \bar{x} = E[x(t)|Z_{k-1}] \), is

\[ x(t + \epsilon) \approx x(t) + \epsilon f(\bar{x}, 0) + F(x(t) - \bar{x}) + Gw(t), \] (80)

where \( F \) and \( G \) are the respective partial derivatives. Then,

\[ E[x(t + \epsilon)|Z_{k-1}] \approx E[x(t)|Z_{k-1}] + \epsilon f(\bar{x}, 0), \] (81)

\[ (E[x(t + \epsilon)|Z_{k-1}] - E[x(t)|Z_{k-1}])/\epsilon \approx f(\bar{x}, 0). \] (82)

The product \( x(t + \epsilon)x(t + \epsilon) \) can be expressed as

\[ x(t + \epsilon)x(t + \epsilon) \approx x(t)x(t) + \epsilon x(t)F(x(t) - \bar{x}) + \epsilon Gw(t) + \epsilon f(\bar{x}, 0)x' + \epsilon F(x(t)x(t)|Z_{k-1} - \bar{x}x') + \epsilon GQ(t)G' + O(\epsilon^2). \] (83)

Hence, the covariance matrix is

\[ \text{P}(t + \epsilon) = E[(x(t + \epsilon)x(t + \epsilon)|Z_{k-1}) - E[x(t + \epsilon)|Z_{k-1}]E[x(t + \epsilon)|Z_{k-1}]
\]

\[ \approx E[(x(t)x(t)|Z_{k-1}) - E[x(t)|Z_{k-1}]E[x(t)|Z_{k-1}]
\]

\[ \approx F(x(t)x(t)|Z_{k-1} - \bar{x}x') + GQ(t)G' + O(\epsilon^2). \] (84)

Next we have

\[ \text{P}(t + \epsilon) - \text{P}(t)/\epsilon \approx \text{P}(t)F' + \text{FP}(t) + GQG' + O(\epsilon). \] (85)

In limits when \( \epsilon \rightarrow 0 \),

\[ \text{P} \approx \text{P}(t)(t)F(t)' + \text{P}(t)(t)' + G(t)Q(t)G(t)' \] (86)

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REFERENCES


